

$$x_1 = 1, \quad x_2 = -(a_1 - \lambda)/b_1,$$

$$x_i = -[b_{i-2}x_{i-2} + (a_{i-1} - \lambda)x_{i-1}]/b_{i-1}, \quad i = 3, \dots, n.$$

However, the above process is not always successful because the last equation $b_{n-1}x_{n-1} + (a_n - \lambda)x_n = 0$ may be strongly violated. The main reason is not the round-off error in computer arithmetic but the slight departure of λ from the true eigenvalue λ_* . Indeed the last equation may be violated even if all operations are done in exact arithmetic but with a slightly inaccurate λ . Wilkinson [6,7] tried to cure this defect but could not succeed. Only in 1983, Godunov [4] found an elegant solution to this problem. A detailed exposition of Godunov’s solution is found, e.g., in [3]. In 1995, Fernando [2] independently proposed his own solution, which is applicable in more general situation and looks slightly simpler than Godunov’s solution.

The present work was initially motivated by an attempt to combine Fernando’s approach with the deflation techniques developed by Godunov and his collaborators, and Sorensen’s implicitly restarted Lanczos [5] was considered among possible applications. In pursuit of this goal a new deflation procedure was discovered, which can be referred to as an “inner deflation”. In the classical QR algorithm, deflation is restricted to one of the ends of the tridiagonal band. However, in the proposed inner deflation the Givens rotations start from both ends of the band and meet inside it. The deflated eigenvalue emerges at the meeting point on the main diagonal of the transformed tridiagonal matrix. The rest of the transformed matrix forms a tridiagonal band with a bulge near the meeting point that can be chased in any direction. The inner deflation is simple and robust and provides an alternative to inverse iteration for computing several eigenvectors of T . The inner deflation idea has previously been outlined in [1, Section 3.5].

2. The inner deflation

Let us denote the QR factorization of $T - \lambda I$ by Q^+R and QL factorization of $T - \lambda I$ by Q^-L . Then

$$Q^+ = G_1^+ G_2^+ \cdots G_{n-1}^+, \quad Q^- = G_{n-1}^- G_{n-2}^- \cdots G_1^-,$$

where the Givens rotations

$$G_i^+ = \begin{pmatrix} I & & & \\ & c_i^+ & s_i^+ & \\ & -s_i^+ & c_i^+ & \\ & & & I \end{pmatrix}, \quad G_i^- = \begin{pmatrix} I & & & \\ & c_i^- & s_i^- & \\ & -s_i^- & c_i^- & \\ & & & I \end{pmatrix}$$

have the blocks

$$\begin{pmatrix} c_i^\pm & s_i^\pm \\ -s_i^\pm & c_i^\pm \end{pmatrix}$$

$$\begin{aligned} \pm \tilde{\Gamma}_k x_k &= e_k^T \tilde{N}_k x \\ &= e_k^T [\tilde{Q}_k^T (T - \lambda I)] x + e_k^T \Delta_k x \\ &= e_k^T \tilde{Q}_k^T (\lambda_* - \lambda) x + e_k^T \Delta_k x, \end{aligned}$$

where $\tilde{\Gamma}_k = |\tilde{N}_k(k, k)|$ is the computed value of Γ_k . Again by choosing k_0 such that $|x_{k_0}| = \max_i |x_i|$ we derive the final estimate

$$\min_i \tilde{\Gamma}_i \leq \tilde{\Gamma}_{k_0} \leq \sqrt{n} |\lambda_* - \lambda| + \sqrt{n} C(n) \epsilon \|T - \lambda I\|_2.$$

4. Numerical example

Consider the positive definite tridiagonal matrix from [4],

$$T = \begin{bmatrix} 2 & 1 & & & \\ 1 & 1 + \rho & \rho & & \\ & \rho & 2\rho & \rho & \\ & & \rho & 1 + \rho & 1 \\ & & & 1 & 2 \end{bmatrix},$$

where $\rho \ll 1$. The smallest eigenvalue of this matrix lies in the interval $(0, 2\rho)$. Setting ρ to machine precision ($\approx 2.22 \times 10^{-16}$ in IEEE double precision arithmetic) and $\lambda = 0$, we find the computed Γ values to be:

$$\tilde{\Gamma} = [.666667 \quad .408248 \quad .444089 \times 10^{-15} \quad .408248 \quad .666667].$$

Thus in this case $k_0 = 3$ and no other value of k allows deflation.

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