Chapter 8: Trigonometric Functions

§8.1: Periodic Functions

In this chapter we'll be studying *trigonometric functions*, which are examples of *periodic functions* — functions that "repeat" their behavior.

Definition 1 (Periodic Functions)**.**
A nonconstant function f is called periodic if there exists some number $P > 0$ such that

$$f(t) = f(t + P) \quad \textit{for all } t \textit{ in the domain.}$$

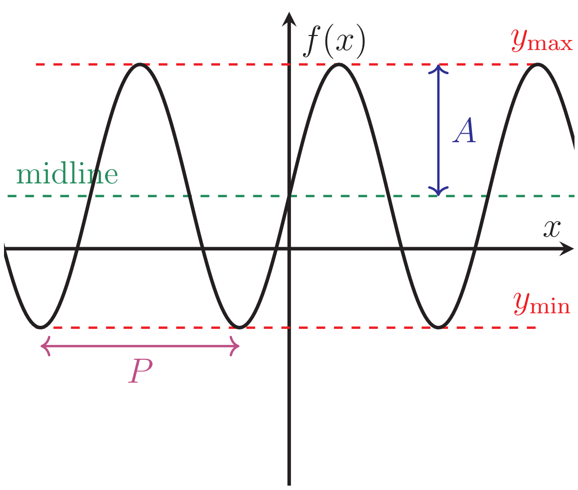
- *The period is the smallest such P, i.e., the time it takes for f to do one cycle.*
- *The amplitude is half the vertical height:*

$$A = \frac{y_{max} - y_{min}}{2}.$$

- *The midline is the horizontal line through the midpoint of the max and min y-values:*

$$y = \frac{y_{max} + y_{min}}{2}.$$

  (The line that the function is symmetric about)



Notice that the amplitude is always a positive number, and the midline is a **function** — namely a horizontal line. So your midline should always be in the form $y = C$ or $f(x) = C$ where C is the number.

To determine the period, it is easiest to choose points that are clearly corresponding parts of the cycle. For instance, measuring from peak-to-peak or valley-to-valley.

§5.3: Numerically Given Exponential Functions

Determining if a numerically given function is *exponential* is very similar to how we determined if it was *linear* in §2.3.

Result 13 (Test for Exponential Functions)**.**
A numerically given function could be exponential if its inputs are ***evenly spaced*** *and the ratio of consecutive outputs is always* ***the same number****.*

Why does this work?

Say we had $f(x) = A \cdot b^x$. Take some evenly spaced x values of x_1, x_2, x_3, x_4 so that

$$x_2 - x_1 = \Delta x, \quad x_3 - x_2 = \Delta x, \quad x_4 - x_3 = \Delta x.$$

Then we have $f(x_1) = A \cdot b^{x_1}$, $f(x_2) = A \cdot b^{x_2}$, $f(x_3) = A \cdot b^{x_3}$, and $f(x_4) = A \cdot b^{x_4}$. Next, take consecutive ratios of outputs and simplify:

$$\begin{aligned} \frac{A \cdot b^{x_2}}{A \cdot b^{x_1}} &= \frac{b^{x_2}}{b^{x_1}} = b^{x_2 - x_1} = b^{\Delta x} \\ \frac{A \cdot b^{x_3}}{A \cdot b^{x_2}} &= \frac{b^{x_3}}{b^{x_2}} = b^{x_3 - x_2} = b^{\Delta x} \\ \frac{A \cdot b^{x_4}}{A \cdot b^{x_3}} &= \frac{b^{x_4}}{b^{x_3}} = b^{x_4 - x_3} = b^{\Delta x} \end{aligned}$$

The consecutive ratios of outputs are equal and they are all equal to $b^{\Delta x}$, which is a constant number![1]

If the given inputs are **not** evenly spaced, it could still be exponential, but we have to "fill in" the missing values and check that Result 13 holds.

Note: Evenly spaced inputs **does not necessarily mean** that the change in inputs is 1. For instance, $2, 3, 4, 5$ and $2, 5, 8, 11$ are evenly spaced (change is 1 and 3, resp.), but $0, 3, 10, 340$ is not.

[1] The other way is true as well. That is, if a function with evenly spaced input has a constant ratio of consecutive outputs, then the function is indeed exponential. Verifying this requires differential equations.

§4.2: Fractional Exponents

Recall that:

- b is a square root of a if $b^2 = a$.
  - Example: 6 is a square root of 36, since $6^2 = 36$. (Is -6 a square root?)
  - "What number do I multiply two copies of to get 36?"
- b is a cube root of a if $b^3 = a$.
  - Example: 2 is a cube root of 8, since $2^3 = 8$. (Is -2 a cube root?)
  - "What number do I multiply three copies of to get 8?"

We can then write

$$\sqrt{36} = \pm 6, \qquad \sqrt[3]{8} = 2.$$

Definition 5 (nth roots)**.**
An nth root of a is the number b satisfying

$$\underbrace{b \times b \times \cdots \times b}_{n \text{ times}} = a, \tag{1}$$

or alternatively, $b^n = a$. We call n the index. If (1) exists, we denote it as

$$\sqrt[n]{a} = b.$$

By convention, $\sqrt{a} = \sqrt[2]{a}$.

When you see $\sqrt[n]{a}$, you should think "what number do I multiply n copies of to get a?"

Example 6.
Compute the following:

1. $\sqrt[3]{27}$
2. $\sqrt[4]{16}$
3. $\sqrt[3]{-27}$
4. $\sqrt[4]{-4}$

Completing the square is another factoring form that easily gives us a lot of information about the quadratic. Similar to factoring, there is a slightly different approach for monic vs non-monic.

Result 14 (Complete square on $x^2 + bx + c$)**.**

1. *Compute* $\left(\frac{b}{2}\right)^2$.
2. *Add and subtract this to the expression:*

$$x^2 + bx + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2.$$

3. *Factor the "left" quadratic into*

$$\left(x + \frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2.$$

Example 15.

Complete the square on the quadratics.

1. $x^2 + 6x + 10$

2. $x^2 - 8x + 1$

3. $x^2 + 5x - 3$

§2.3: Numerically Given Linear Functions

Recall Example 4 where we showed that linear functions have an average rate of change that is constant (meaning equal) over *any* interval.

> **Main Result:**
> A numerically given function is linear if it has an average rate of change that is the same on any interval.
> In other words, if the slope between consecutive points is always the same.

Given a numerical function (described by a table):

1. Find the change in consecutive inputs. Δx
2. Find the change in consecutive outputs. $\Delta f(x)$
3. Determine the slope for these points. $\dfrac{\Delta f(x)}{\Delta x}$

If **ALL** the numbers from 3 are equal, then the function is linear.

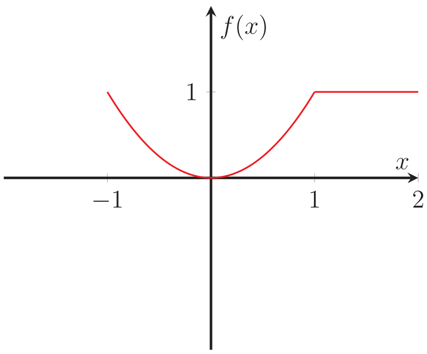
Example 15.
Determine if the numerically given functions are linear.

x	0	1	2	3	4	5
$f(x)$	4	4.5	5	5.5	6	6.5

x	2	4	6	8	10
$f(x)$	6	4.6	3.2	1.8	0.4

Definition 17 (Increasing and Decreasing Functions)**.**
A function is increasing on an interval if its y-values (output values) ***increase*** *as its x-values (input values) increase.*
A function is decreasing on an interval if its y-values (output values) ***decrease*** *as its x-values (input values) increase.*

Example 18.
Determine the intervals where the function is increasing or decreasing.



Just because we can draw a curve on the xy-plane does not make the curve a function!

Result 19 (Vertical Line Test)**.**
If there exists a vertical line that intersects the curve more than once, then the curve ***is not*** *a function. If every vertical line intersects the curve only once, then the curve* ***is*** *a function.*