Matrix Products Coinciding with Concatenation

Kyle Monette

Department of Mathematics and Applied Mathematical Sciences
University of Rhode Island (CU '22 Alum)

MCCNNY

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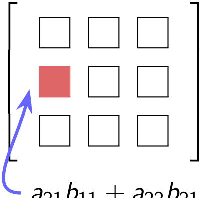
Matrix Multiplication

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$



$$a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

Introduction

Our interests include:

- Matrices where matrix multiplication is elementwise concatenation.

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- Matrices where matrix multiplication is matrix addition.

$$\begin{bmatrix} 4 & 8 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} -4 & 8 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 16 \\ 4 & 4 \end{bmatrix} \qquad AB = A + B$$

Introduction

Notation:

- Let $N_d = \{n \in \mathbb{N} \mid n \text{ has } d \text{ digits}\}$, that is, $N_1 = \{1, \ldots, 9\}$ and $N_2 = \{10, \ldots, 99\}$.

- Let $N_d^{n\times n}$ be the set of $n \times n$ matrices with entries in N_d.

- Let $Z_d = \{n \in \mathbb{Z} \mid n \text{ has } d \text{ digits}\}$ and $Z_d^{n\times n}$ the set of $n \times n$ matrices with entries in Z_d.

- A set $\{A_1, \ldots, A_k\}$ of matrices in $N_d^{n\times n}$ is said to satisfy the multiplication concatenation property (MCP) if

$$A_1 \cdot \ldots \cdot A_k \quad = \quad 10^{d(k-1)}A_1 + 10^{d(k-2)}A_2 + \cdots + 10^d A_{k-1} + A_k.$$

- A set $\{A_1, \ldots, A_k\}$ of matrices in Z_d^{nxn} is said to satisfy the multiplication addition property (MAP) if

$$A_1 \cdot \ldots \cdot A_k \quad = \quad A_1 + A_2 + \cdots + A_{k-1} + A_k.$$

Case 1: Single MCP Matrices

First, consider matrices $A \in N_d^{n \times n}$ that satisfy the MCP by themselves:

$$A \cdot A = 10^d A + A \quad \Leftrightarrow \quad A^2 = (10^d + 1)A.$$

For example,

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^2 = \begin{bmatrix} 33 & 44 \\ 66 & 88 \end{bmatrix} \qquad A^2 = 11A$$

$$\begin{bmatrix} 60 & 30 \\ 82 & 41 \end{bmatrix}^2 = \begin{bmatrix} 6060 & 3030 \\ 8282 & 4141 \end{bmatrix} \qquad A^2 = 101A$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{bmatrix}^2 = \begin{bmatrix} 11 & 33 & 11 \\ 33 & 99 & 33 \\ 11 & 33 & 11 \end{bmatrix} \qquad A^2 = 11A$$

Case 1: Single MCP Matrices

Matrices $A \in N_d^{n \times n}$ such that $A^2 = (10^d + 1)A$ have the following properties:

- Eigenvalues are 0 and $10^d + 1$
- The minimal polynomial is $m(x) = x^2 - (10^d + 1)x$
- A is diagonalizable
- A has determinant 0 and trace $10^d + 1$ (sufficient and necessary)

Future Work

Some possible directions for the future:

- Is there a size limit for the matrices satisfying $A^2 = (10^d + 1)A$?
- Can we find 5×5 or larger matrices satisfying $A^2 = (10^d + 1)A$?
- Can we think about *solving* a matrix equation like the ones in the MCP or MAP?
- Are there matrices B so that $\{A_1, \ldots, A_k, B\}$ **cannot** satisfy the MCP or MAP?
- Study the connection to *right quasiregular elements* from abstract algebra.

Questions?