Double Pendulum: Lagrangian Mechanics and Chaos

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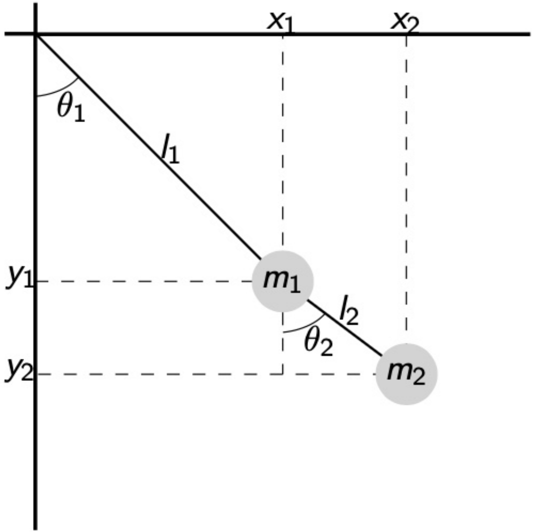
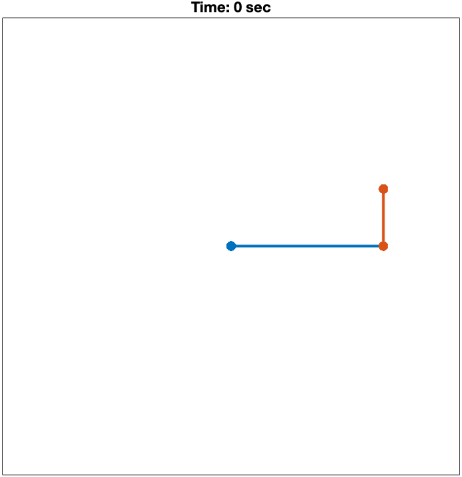
Overview

The Double Pendulum

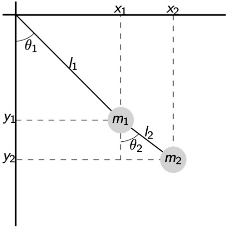


Animation



$$\theta_1(0) = \pi/2,\ \theta_2(0) = \pi,\ \dot{\theta_1} = \dot{\theta_2} = 0,\ l_1 = 4,\ l_2 = 1.5,\ m_1 = m_2 = 1$$

Equations of Motion



$$x_1 = l_1 \sin(\theta_1) \qquad \dot{x}_1 = l_1 \cos(\theta_1)\dot{\theta}_1$$

$$x_2 = l_1 \sin(\theta_1) + l_2 \sin(\theta_2) \qquad \dot{x}_2 = l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_2)\dot{\theta}_2$$

$$y_1 = -l_1 \cos(\theta_1) \qquad \dot{y}_1 = l_1 \sin(\theta_1)\dot{\theta}_1$$

$$y_2 = -l_1 \cos(\theta_1) - l_2 \cos(\theta_2) \qquad \dot{y}_2 = l_1 \sin(\theta_1)\dot{\theta}_1 + l_2 \sin(\theta_2)\dot{\theta}_2$$

This gives the kinetic and potential energy

$$T = \frac{1}{2} m_1 (\dot{x}_1{}^2 + \dot{y}_1{}^2) + \frac{1}{2} m_2 (\dot{x}_2{}^2 + \dot{y}_2{}^2)$$

$$U = m_1 g y_1 + m_2 g y_2$$

Lagrangian System

In terms of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$,

$$T = \frac{m_1 + m_2}{2} l_1^2 \dot{\theta_1}^2 + \frac{m_2}{2} l_2^2 \dot{\theta_2}^2 + m_2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} \cos(\theta_1 - \theta_2)$$

$$U = -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2)$$

We define the *Lagrangian* $\mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, t) = T - U$:

$$\mathcal{L}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \frac{m_1 + m_2}{2} l_1^2 \dot{\theta_1}^2 + \frac{m_2}{2} l_2^2 \dot{\theta_2}^2 + m_2 l_1 l_2 \dot{\theta_1} \dot{\theta_2} \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2)$$

Lagrangian System

We now make use of the *Euler-Lagrange Equations*

$$\frac{d}{dt}\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}}_{p_i} - \frac{\partial L}{\partial \theta_i} = 0, \quad i = 1, 2.$$

Hence our Lagrangian ODE system

$$\begin{aligned} & l_1(m_1 + m_2)\ddot{\theta_1} + m_2 l_2 \ddot{\theta_2}\cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta_2}^2 \sin(\theta_1 - \theta_2) \\ & \qquad + (m_1 + m_2) g \sin(\theta_1) = 0 \\ & l_2\ddot{\theta_2} + l_1\ddot{\theta_1}\cos(\theta_1 - \theta_2) + l_1\dot{\theta_1}^2 \sin(\theta_1 - \theta_2) + g\sin(\theta_2) = 0 \end{aligned}$$

Hamiltonian System

We define the *Hamiltonian* to be

$$\begin{aligned}\mathcal{H}(\boldsymbol{p}, \boldsymbol{\theta}, t) &= \sum_{i=1}^{n} p_i \dot{\theta}_i - \mathcal{L}, \qquad p_i = \frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} \\ &= T + U.\end{aligned}$$

For the double pendulum,

$$\begin{aligned}\mathcal{H}(\boldsymbol{p}, \boldsymbol{\theta}) = \frac{m_1 + m_2}{2} l_1^2 p_1^2 + \frac{m_2}{2} l_2^2 p_2^2 + m_2 l_1 l_2 p_1 p_2 \cos(\theta_1 - \theta_2) \\ - (m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2)\end{aligned}$$

(substitute $\dot{\theta}_i$ for p_i)

Hamiltonian System

The Lagrangian system (dimension n) is equivalent to the *Hamiltonian system* (dimension $2n$), given by

$$\dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{\theta}}, \qquad \dot{\boldsymbol{\theta}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}$$

$$\begin{aligned}
\dot{p}_1 &= l_1 l_2 m_2 p_1 p_2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin(\theta_1) \\
\dot{p}_2 &= -l_1 l_2 m_2 p_1 p_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin(\theta_2) \\
\dot{\theta}_1 &= p_1 l_1^2 (m_1 + m_2) + l_1 l_2 m_2 \cos(\theta_1 - \theta_2) \\
\dot{\theta}_2 &= p_2 l_2^2 m_2 + l_1 l_2 m_2 p_1 \cos(\theta_1 - \theta_2)
\end{aligned}$$

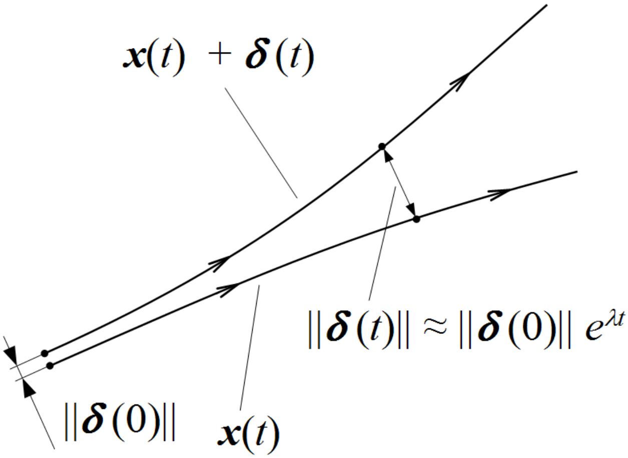
What is Chaos?

Chaos is aperiodic long-term behavior in a deterministic system that exhibits dependence on initial conditions.

-Steven Strogatz

1. "Aperiodic long-term behavior:" Existence of trajectories that do not gravitate to fixed points or periodic orbits.
2. "Deterministic:" Not unpredictable!
3. "Sensitive dependence on initial conditions:" Neighboring trajectories separate exponentially (positive Lyapnuov exponent).

Lyapunov Exponents



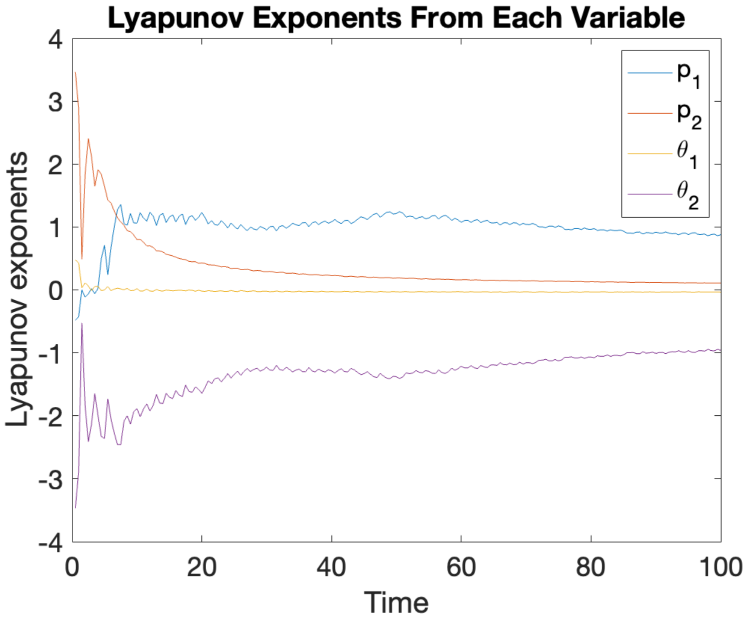
Source: Wikipedia – Lyapunov Exponents

Lyapunov Exponents

A numerical scheme (J.C. Sprott):

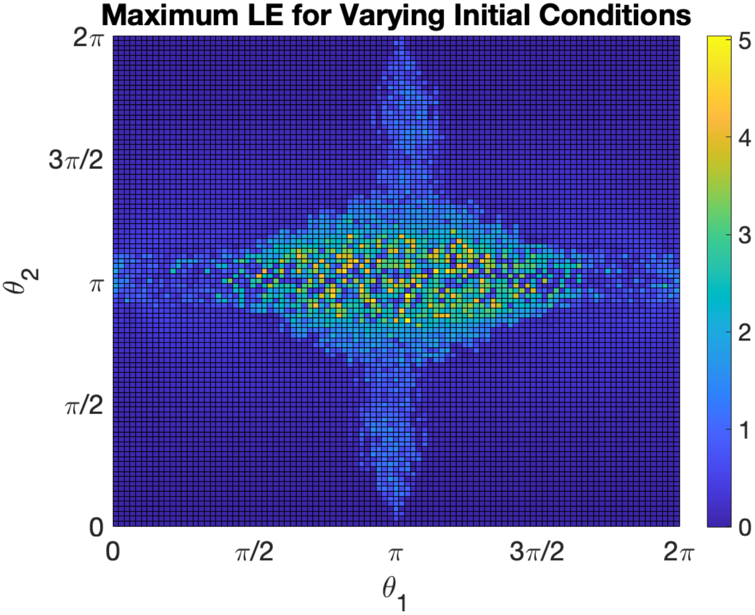
1. Choose an initial condition and a nearby point with separation $\delta(0)$.
2. Iterate once and calculate the new separation $\delta(1)$.
3. Evaluate $\ln(\delta(1)/\delta(0))$.
4. Readjust one orbit so its separation is $\delta(0)$ in the same direction as $\delta(1)$.
5. Repeat steps 2-4, and compute the average of step 3.
6. The LE is
$$\frac{\ln \frac{\delta(1)}{\delta(0)}}{\Delta t}.$$

Results

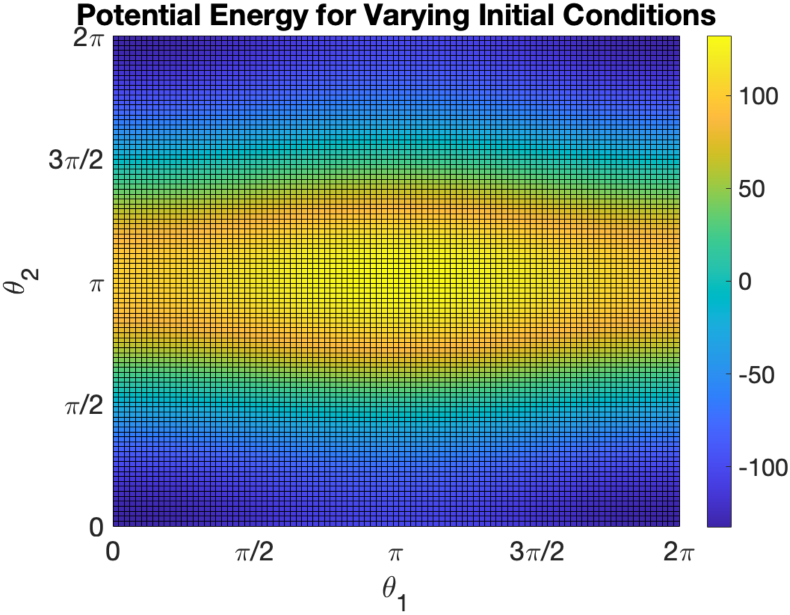


Lyapunov Exponents From Each Variable
Lyapunov exponents
Time
4
3
2
1
0
-1
-2
-3
-4
0
20
40
60
80
100
p_1
p_2
θ_1
θ_2

Results


Maximum LE for Varying Initial Conditions
2π
$3\pi/2$
π
$\pi/2$
0
θ_2
0
$\pi/2$
π
$3\pi/2$
2π
θ_1
5
4
3
2
1
0

Results



Potential Energy for Varying Initial Conditions
2π
3π/2
π
π/2
0
θ2
0
π/2
π
3π/2
2π
θ1
100
50
0
-50
-100

References

- Alligood, Sauer, Yorke, “Chaos: An Introduction to Dynamical Systems”.
- Vasiliy Govorukhin (2022). Calculation Lyapunov Exponents for ODE (https://www.mathworks.com/matlabcentral/fileexchange/4628-calculation-lyapunov-exponents-for-ode), MATLAB Central File Exchange. Retrieved March 22, 2022.
- J.C. Sprott (1997). Numerical Calculation of Largest Lyapunov Exponent (https://test-sprott.physics.wisc.edu/chaos/lyapexp.htm). Retrieved March 21, 2022.
- Strogatz, “Nonlinear Dynamics and Chaos”.
- A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, “Determining Lyapunov Exponents from a Time Series,” Physica D, Vol. 16, pp. 285-317, 1985.