PROBLEM STATEMENT

Consider the following iterated integral $\int_0^1 \int_{y^2}^1 y\sin(x^2)\,dxdy$.

(a) Sketch the region of integration.

(b) Evaluate the given integral by first reversing the order of integration.

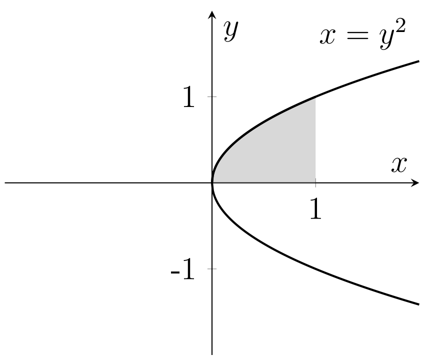
SOLUTION

(a) From the inner integral and outer integral, respectively, we have

$$y^2 \le x \le 1 \qquad \text{condition (1)}$$

$$0 \le y \le 1 \qquad \text{condition (2)}$$

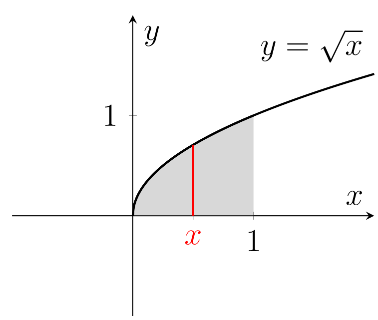
The shaded region below satisfies condition (1) and condition (2).



Note that the region *below* the x-axis and bounded by the curve $x = y^2$ does *not* satisfy condition (1).

(b)

$$\begin{aligned}
\int_0^1 \int_{y^2}^1 y\sin(x^2)\,dx\,dy &= \int_0^1 \int_0^{\sqrt{x}} y\sin(x^2)\,dy\,dx \\
&= \int_0^1 \left(\frac{y^2}{2}\sin x^2\right)\Big|_0^{\sqrt{x}}\,dx \\
&= \int_0^1 \frac{1}{2}x\sin(x^2)\,dx \qquad u = x^2, du = 2xdx \\
&= \frac{1}{4}\int_0^1 \sin(u)\,du \\
&= \frac{1}{4}(-\cos(u))\Big|_0^1 \\
&= -\frac{1}{4}(\cos(1) - \cos(0)) \\
&= \boxed{\frac{1}{4} - \frac{1}{4}\cos(1)}
\end{aligned}$$



PROBLEM STATEMENT

Evaluate the following iterated integral

$$\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx\,.$$

SOLUTION

$$\begin{aligned}
\int_1^2 \int_0^{x^2} \frac{1}{x+y}\, dy\, dx &= \int_1^2 \ln|x+y|\Big|_0^{x^2} dx = \int_1^2 \big(\ln(x+x^2) - \ln(x+0)\big)\, dx \\
&= \int_1^2 \ln\left(\frac{x+x^2}{x}\right) dx \\
&= \int_1^2 \ln(1+x)\, dx \\
&= (x+1)\ln(x+1) - x\Big|_1^2 \qquad \text{(see below for this step)} \\
&= \big((1+1)\ln(2+1) - 2\big) - \big((1+1)\ln(1+1) - 1\big) \\
&= 3\ln(3) - 2 - 2\ln(2) + 1 \\
&= \boxed{3\ln(3) - 2\ln(2) - 1}
\end{aligned}$$

To compute $\int \ln(1+x)\, dx$, use integration by parts:

$$u = \ln(1+x) \qquad du = \frac{1}{x+1} dx$$

$$v = x \qquad dv = dx$$

Then we obtain

$$\begin{aligned}
\int \ln(1+x)\, dx &= x\ln(1+x) - \int \frac{x}{x+1}\, dx \\
&= x\ln(1+x) - \int \frac{x+1-1}{x+1}\, dx \\
&= x\ln(1+x) - \int \left(1 - \frac{1}{x+1}\right) dx \\
&= x\ln(1+x) - x + \ln(x+1) + C \\
&= (x+1)\ln(x+1) - x + C
\end{aligned}$$

(c) $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -4 & 5 & -1 \end{bmatrix}$

Solution: The characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -4 & 5 & -1-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)(-1-\lambda) \end{aligned}$$

Setting this polynomial to 0, then the eigenvalues of A are

$$\lambda = 3, \quad 2, \quad -1.$$

$\boxed{\lambda = 3}$ We need to solve $(A - 3I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -4 & 5 & -4 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

As x_3 is a free variable, let $x_3 = \alpha$. Then

$$x_2 - 4x_3 = 0 \quad \Longrightarrow \quad x_2 = 4x_3 = 4\alpha$$

and similarly

$$x_1 - 4x_3 = 0 \quad \Longrightarrow \quad x_1 = 4x_3 = 4\alpha.$$

An eigenvector is

$$\vec{x} = \begin{bmatrix} 4\alpha \\ 4\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

and so a basis for $E_{\lambda=3}$ is

$$\left\{ \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

$\boxed{\lambda = 2}$ We need to solve $(A - 2I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -4 & 5 & -3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

As x_3 is a free variable, let $x_3 = \alpha$. Then

$$x_2 - \frac{3}{5}x_3 = 0 \quad \Longrightarrow \quad x_2 = \frac{3}{5}x_3 = \frac{3}{5}\alpha.$$

An eigenvector is

$$\vec{x} = \begin{bmatrix} 0 \\ 3/5\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 3/5 \\ 1 \end{bmatrix}$$

Chapter 8: Trigonometric Functions

§8.1: Periodic Functions

In this chapter we'll be studying *trigonometric functions*, which are examples of *periodic functions* — functions that "repeat" their behavior[1].

Definition 1 (Periodic Functions)**.**
A nonconstant function f is called periodic if there exists some number $P > 0$ such that

$$f(t) = f(t+P) \quad \text{for all } t \text{ in the domain.}$$

- *The period is the smallest such P, i.e., the time it takes for f to do one cycle.*
- *The amplitude is half the vertical height:*

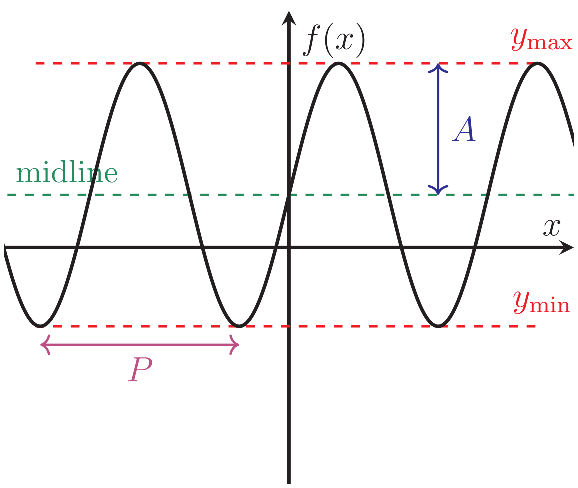
$$A = \frac{y_{max} - y_{min}}{2}.$$

- *The midline is the horizontal line through the midpoint of the max and min y-values:*

$$y = \frac{y_{max} + y_{min}}{2}.$$

(*The line that the function is symmetric about*)

The units of amplitude are the units of the output variable, and the units of the period are the units of the input variable.



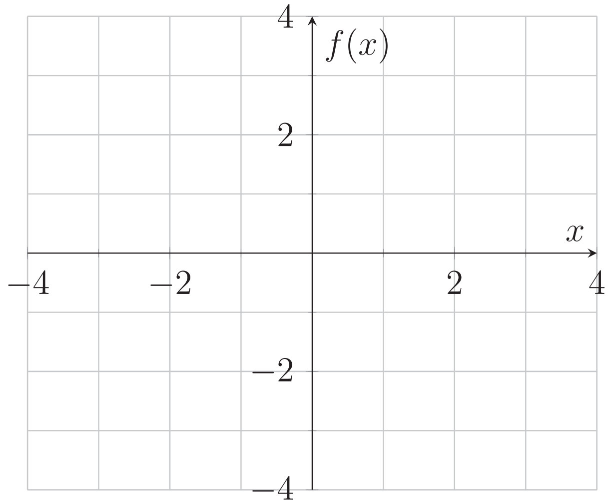
Notice that the amplitude is always a positive number, and the midline is a **function** — namely a horizontal line. So your midline should always be in the form $y = C$ or $f(x) = C$ where C is the number.

To determine the period, it is easiest to choose points that are clearly corresponding parts of the cycle. For instance, measuring from peak-to-peak or valley-to-valley.

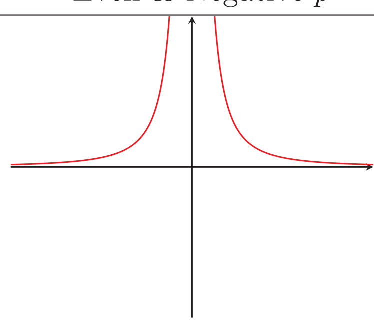
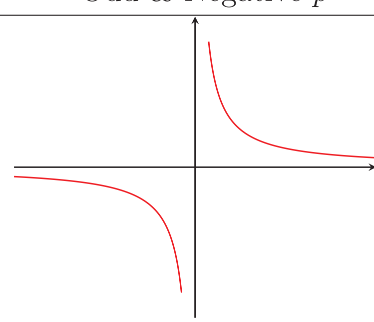
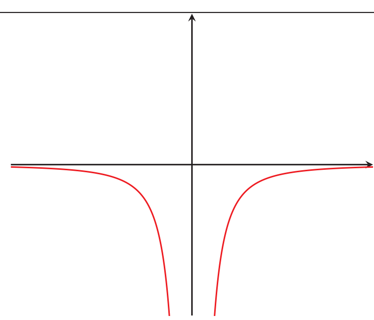
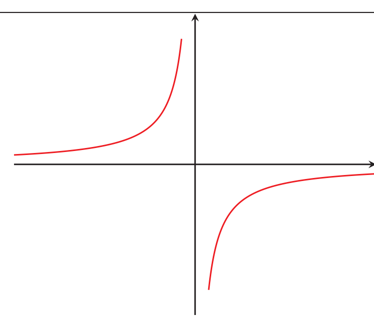
[1]There are periodic functions that *are not* trigonometric, but are rare and not that useful.

Negative Even Exponents

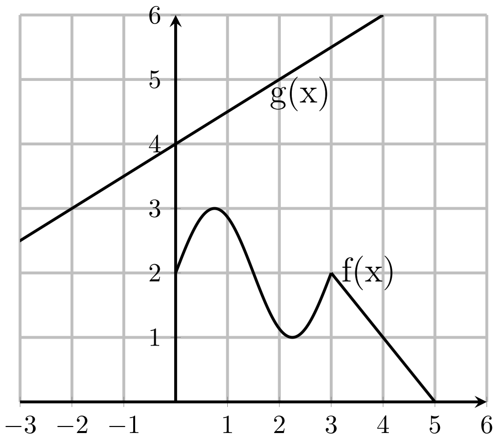
x	$f(x) = \frac{1}{x^2}$
-2	
-1	
0	
1	
2	
$-\frac{1}{2}$	
$\frac{1}{2}$	
$-\frac{1}{4}$	
$\frac{1}{4}$	



Negative Integer Exponents

$f(x) = kx^p$	Even & Negative p	Odd & Negative p
Positive coefficient $k > 0$		
Negative coefficient $k < 0$		

13. The graphs of functions $f(x)$ and $g(x)$ are given below. Compute the following values. *Hint: Do not spend time trying to determine the formulas for the functions!*



(a) $(g \circ f)(0) = \underline{\quad 5 \quad}$

(b) $(f \circ g)(2) = \underline{\quad 0 \quad}$

(c) $(g \circ f)(5) = \underline{\quad 4 \quad}$

(d) $(f \circ g)(-2) = \underline{\quad 2 \quad}$

(e) $(f \circ g)(0) = \underline{\quad 1 \quad}$

14. What is the exact value of $\log_6\left(\frac{1}{\sqrt[5]{6}}\right)$?

A. $\frac{1}{6}$

B $\frac{-1}{5}$

C. 6

D. $-\frac{1}{6}$

15. Decompose the function $y = \frac{1}{\sqrt{x+3}}$ into two functions $f(x)$ and $g(x)$ so that $y = f(g(x))$. *Do not choose $f(x)$ nor $g(x)$ to be the function x.*

Solution:

There is more than one correct answer here — below are some of them.

$$g(x) = x + 3 \quad \text{and} \quad f(x) = \frac{1}{\sqrt{x}}$$

$$g(x) = \sqrt{x+3} \quad \text{and} \quad f(x) = \frac{1}{x}$$

QR Factorization

- $A \in \mathbb{R}^{m \times n}$, $m \geq n$, with linearly independent columns has a unique reduced QR factorization $A = \widehat{Q}_{m \times n} \cdot \widehat{R}_{n \times n}$.

- … and also has a full QR factorization

$$A = Q_{m \times m} \cdot R_{m \times n} = \left[\begin{array}{c|c} \widehat{Q}_{m \times n} & \widetilde{Q}_{m \times (m-n)} \end{array} \right] \left[\begin{array}{c} \widehat{R}_{n \times n} \\ \hline 0_{(m-n) \times n} \end{array} \right]$$

where Q is orthogonal and R has positive diagonal entries.